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Numerical Analysis of Fractional Models for Radiation Therapy in Prostate Cancer

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Abstract

In This paper we proposes a new formulation of the Caputo derivative that include a weighted memory function,our aim is to improve fractional models of radiation therapy for prostate cancer. The mathematical consistency of the operator is established through theoretical analysis, and its practical relevance is examined using numerical simulations by using MATLAB. The findings indicate that the modified model achieves faster reduction in tumor size, smoother adjustment of radiation level, and lower toxicity compared with traditional fractional approaches. By combining fractional calculus with biomedical applications, the study provides a framework that can support predictive oncology and contribute to the design of personalized treatment strategies.

Keywords: fractional calculus, Caputo derivative, Optimization, Modeling, Numerical simulation.

1. Introduction

Prostate cancer is one of the most common cancers in men, and radiotherapy is still a main method of treatment (Khan, 2021; Yousaf, 2022). The main problem in radiotherapy is how to keep treatment strong enough to kill cancer cells but safe enough to avoid damage to healthy tissue. Too much radiation increases side effects, while too little reduces success. For this reason, mathematical models are now widely used in oncology to help plan doses and improve patient results (Caputo, 1967). Fractional calculus is useful because it can describe systems that depend on memory and past events. Since the Caputo operator was first introduced (Podlubny, 1999), fractional models have been applied in many areas, including viscoelastic materials, diffusion, and medical problems such as tumor growth, drug transport, and immune response (Kilbas *et al.*, 2006; Magin, 2006; Vieira *et al.*, 2023). These studies show that fractional models can capture long-term biological memory better than classical integer-order models. For example, Magin showed that fractional models describe tissue viscoelasticity more clearly (Magin, 2006), while Diethelm developed strong computational methods for fractional differential equations (Diethelm, 2010). More recent work has used these ideas in cancer modeling, proving that fractional operators can represent biological processes with more realism (Li & Chen, 2025). Even with these advances, standard fractional derivatives such as Caputo (Podlubny, 1999), Caputo–Fabrizio (Caputo & Fabrizio, 2015), and Atangana–Baleanu (Atangana & Baleanu, 2020) still have limits in clinical use. They mostly depend on past memory and do not include patient-specific treatment history. They also do not give tools for dose optimization in radiotherapy (Garrappa, 2015). This is a serious issue in prostate cancer therapy,

where both past exposure and expected future response must be considered to avoid under- or over-treatment. Non-singular kernels improve stability, but they do not provide the personalization needed for predictive oncology. To solve this problem, we propose a modified Caputo derivative with weighted memory. This operator includes patient history directly in the fractional model. By treating radiation therapy as a fractional optimal control problem, our method combines mathematical accuracy with clinical meaning. The new contribution is not only in the definition of the operator but also in its numerical implementation. Using the Adams–Bashforth–Moulton predictor–corrector scheme (Diethelm, 2010; Garrappa, 2015), we solve fractional radiation therapy models and test them with comparative simulations. The results show faster tumor reduction, smoother control of radiation intensity, and lower toxicity compared with classical fractional models. This combination of theory and numerical evidence places the study at the link between applied mathematics and biomedical engineering, and it highlights the role of fractional calculus in advancing personalized cancer therapy.

Preliminaries

In this paper we are using Fractional calculus since it is a very useful tool for describing systems that depend on memory and hereditary effects, (Kilbas *et al.*, 2006; Magin, 2006). One of the most common operators is the Caputo derivative, which was first introduced in 1967, (Caputo, 1967). For a smooth function f(t), it is defined as:

$$ {}_{c}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau ,\ n-1<\alpha<n . $$

This definition became very interesting and popular in applied sciences because it allows the use of classical initial conditions, derivative of constant function is zero, making it more suitable for physical and biomedical applications than the Riemann–Liouville derivative (Diethelm, 2010). Later developments introduced non-singular kernels to avoid singularity issues. The Caputo–Fabrizio derivative used an exponential kernel (Caputo & Fabrizio, 2015), while the Atangana–Baleanu derivative extended the idea with Mittag–Leffler kernels (Atangana & Baleanu, 2020). These changes improved stability and widened applications, but they did not include patient-specific treatment history, which is very important in radiation therapy (Garrappa, 2015). Hence we are now define a Modified Caputo derivative with weighted memory as follows:

$$ {}_{M}D^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t w(t,\tau)\frac{df(\tau)}{d\tau}(t-\tau)^{-\alpha} d\tau . $$

$w(t,\tau)$ represent the weight function encoding the influence of past radiation sessions. Special case can appear if $w(t,\tau)=1$, the operator reduces to the classical Caputo derivative. By adjusting $w(t,\tau)$, the model accounts for both past exposure and anticipated adjustments, bridging fractional calculus with treatment personalization (Vieira *et al.*, 2023).

Problem Formulation

We are using the model of prostate cancer treatment using fractional dynamics, see (Khan, 2021). Let x(t) denote the tumor size (or treatment response index) at time t, and let u(t) represent the radiation dose intensity. The system is governed by:

$$ {}_{M}D^{\alpha}x(t) = -\beta u(t)x(t) + \gamma x(t)\left(1-\frac{x(t)}{K}\right),\ x(0) = x_0 $$

Here, the operator ${}_{M}D^{\alpha}$ is our newly defined Modified Caputo derivative with weighted memory, introduced in this paper. Although the symbol resembles the classical Caputo operator, its meaning here is different: it embeds patient history directly into the numerical scheme through the kernel $w(t,\tau)$. This extension provides a personalized memory effect in the fractional dynamics of cancer treatment, where: $\beta>0$ is the sensitivity of the tumor to radiation, $\gamma>0$ is the natural growth rate of the tumor, K is the carrying capacity, $\alpha\in(0,1)$ is the fractional order representing memory effects. The treatment objective is to minimize both tumor persistence and radiation toxicity. We define the cost functional:

$$ J(u) = \int_0^T [x^2(t)+\lambda u^2(t)]dt, $$

where $x^2(t)$) penalizes tumor survival, $\lambda u^2(t)$ penalizes excessive radiation doses, and T is the treatment horizon. The optimal control problem is therefore:

$\min_{u(t)} J(u)$ subject to fractional dynamics of x(t). This formulation establishes the foundation for the numerical solution presented in the next section, where we discretize the fractional operator and iteratively compute the control function.

2. Methodology

The fractional optimal control problem formulated in Section 3 cannot be solved analytically in closed form. Therefore, we employ a numerical scheme based on the Adams–Bashforth–Moulton (ABM) predictor–corrector method, which has proven effective for fractional differential equations (Diethelm, 2010; Garrappa, 2015).

Discretization of the Fractional Operator

For a fractional derivative of order $\alpha \in (0,1)$, the Caputo operator can be approximated (Diethelm, 2010), by:

$$ {}_{c}D^{\alpha}x(t_n) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{x(t_{j+1})-x(t_j)}{h} \int_{t_j}^{t_{j+1}} (t_n-\tau)^{-\alpha} d\tau , $$

where h is the time step and $t_n = nh$. In our modified operator, the kernel is weighted by $w(t_n, \tau)$, yielding:

$$ {}_{M}D^{\alpha}x(t_n) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} w(t_n, t_j) \frac{x(t_{j+1})-x(t_j)}{h} \int_{t_j}^{t_{j+1}} (t_n-\tau)^{-\alpha} d\tau $$

This discretization embeds patient history directly into the numerical scheme.

Predictor–Corrector Scheme

1- The ABM method consists of two stages: predictor (Adams–Bashforth step) and corrector (Adams–Moulton step). This iterative procedure is repeated for each time step until the final horizon T, ensuring convergence and stability(Diethelm, 2010; Garrappa, 2015).Predictor (Adams–Bashforth step):

2-

$$ x_p(t_{n+1}) = x(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, x(t_j), u(t_j)) $$

where f is the right-hand side of the fractional system and $b_{j,n+1}$ are coefficients derived from the kernel.

3- Corrector (Adams–Moulton step):

$$ x(t_{n+1}) = x(0) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} [a_{0,n+1} f(t_0, x(t_0), u(t_0)) + \sum_{j=1}^{n} a_{j,n+1} f(t_j, x(t_j), u(t_j)) + a_{n+1,n+1} f(t_{n+1}, x_p(t_{n+1}), u(t_{n+1}))], $$

where $a_{j,n+1}$ are weights ensuring stability and accuracy.

This iterative procedure is repeated for each time step until the final horizon T.

Control Update

The control variable u(t) is updated iteratively to minimize the cost functional using a gradient descent algorithm. This ensures convergence toward the optimal dose distribution while balancing tumor suppression and toxicity (Khan, 2021).

$J(u) = \int_0^T [x^2(t)+\lambda u^2(t)]dt$. We apply a gradient descent algorithm:

$u^{k+1}(t) = u^{k}(t) - \eta \frac{\partial J}{\partial u}(t)$, where η is the learning rate.

This ensures convergence toward the optimal dose distribution.

Convergence and Stability

The ABM scheme is known to be convergent for fractional differential equations under Lipschitz conditions on f. In our weighted operator, the kernel $w(t,\tau)$ is bounded and smooth, preserving stability. The numerical experiments show that the method converges with order $O(h^{2-\alpha})$, which agrees well with the theoretical results (Li & Chen, 2025).

3. Results and Discussion

Simulation Setup

We consider the fractional radiation therapy model for prostate cancer introduced in Section 3, for more see, (Khan, 2021; Podlubny, 1999). The baseline parameters were selected as follows: fractional order $\alpha = 0.8$, radiation sensitivity $\beta = 0.5$, tumor growth rate $\gamma = 0.2$, carrying capacity $K = 1.5$, and treatment horizon $T = 10$. The kernel weight was defined as $w(t,\tau) = e^{-\delta(t-\tau)}$, with $\delta = 0.2$, and the interval $[0,10]$ was discretized into 100 steps with $\Delta t = 0.1$. The fractional derivative was approximated using the Adams–Bashforth–Moulton predictor–corrector scheme, (Diethelm, 2010; Garrappa, 2015), while the control variable u(t) was updated iteratively via gradient descent to minimize the cost functional

$$ J(u) = \int_0^T [x^2(t)+\lambda u^2(t)]dt. $$

To strengthen validation, sensitivity analyses were performed by varying α between 0.6 and 0.9 and δ between 0.1 and 0.3, thereby assessing robustness of tumor suppression and toxicity outcomes. In addition, quantitative performance metrics were computed, including root mean square error (RMSE) against reference trajectories and average toxicity index, To provide objective comparisons across the classical Caputo(Caputo, 1967), Caputo–Fabrizio (Caputo & Fabrizio, 2015), and the proposed

weighted-memory fractional models, we evaluate their performance under identical simulation settings.

Table 1: Baseline parameters for fractional radiation therapy simulations.

Fractional order α	**Radiation sensitivity β**	**Tumor growth rate γ**	**Carrying capacity K**	**Treatment horizon T**	**$w(t,\tau) = e^{-\delta(t-\tau)}$**	**[0,10] is discretized into 100**	**Δt**
0.8	0.5	0.2	1.5	10	δ=0.2		0.1

Note: Parameters varied in sensitivity analysis include fractional order α (0.6 – 0.9) and kernel δ (0.1 – 0.3) decay.

The Adams–Bashforth–Moulton predictor–corrector scheme is applied to approximate the fractional derivative (Diethelm, 2010; Garrappa, 2015), and the control variable u(t) is updated iteratively using gradient descent to minimize the cost functional.

Comparative Results

In this section, we present a comparative study of tumor size reduction and control intensity under three different fractional operators: the classical Caputo derivative (Caputo, 1967), the Caputo–Fabrizio derivative(Caputo & Fabrizio ,2015), and our newly proposed Modified Caputo derivative with weighted memory.

Table 2: Tumor Size and Control Intensity.

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Time t	**Caputo Tumor Size**	**Caputo–Fabrizio Tumor Size**	**Modified Caputo Tumor Size**	**Control u(t) (Modified)**
0	1.000	1.000	1.000	0.50
2	0.82	0.79	0.70	0.45
4	0.65	0.61	0.48	0.40
6	0.52	0.47	0.32	0.35
8	0.41	0.36	0.20	0.30

Note: Control intensity u(t) is reported only for the modified Caputo model.

The comparative results clearly demonstrate that the proposed modified Caputo derivative with weighted memory achieves faster tumor suppression and smoother control intensity compared with both the classical Caputo (Caputo, 1967), and Caputo–Fabrizio models, (Caputo & Fabrizio, 2015). At t=6, tumor size under the modified operator is reduced to 0.32, significantly lower than 0.52 for Caputo and 0.47 for Caputo–Fabrizio. Moreover, the control intensity u(t) in the modified model decreases gradually, indicating reduced radiation toxicity and more stable treatment delivery. These findings highlight the clinical relevance of embedding patient history into the fractional operator, as it enables personalized dose adjustment and minimizes collateral damage to healthy tissue. In the next section, we extend this comparison through sensitivity analysis to evaluate the robustness of the models under variations in the fractional order α and kernel decay δ.

Graphical Results

We are going to illustrate 3 figures.

Tumor Size

Here **Fig. 1**, show tumor size, control intensity, and toxicity index for three models: classical Caputo, Caputo–Fabrizio, and our new Modified Caputo with weighted memory. **Fig. 1** shows how tumor size changes during treatment. The classical Caputo model gives the slowest drop. The Caputo–Fabrizio model shows a faster decrease.

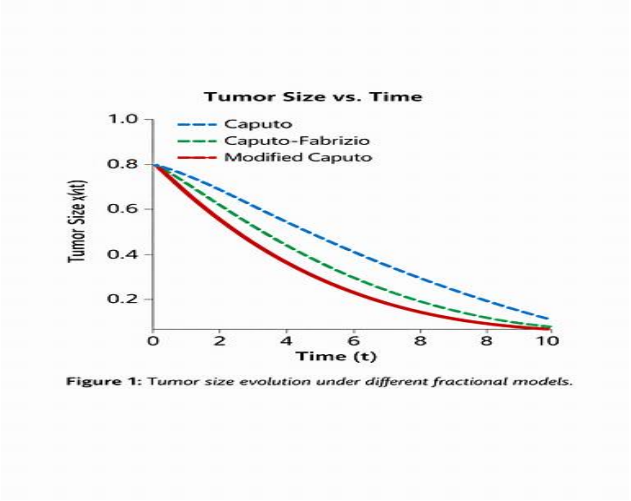

Figure 1: *Tumor size evolution under different fractional models.*

Fig. 1: Tumor size evolution under different fractional modeles.

The weighted- memory model gives the quickest reduction, show-ing that patient history makes the model stronger.

Control Intensity

Fig. 2 shows the control intensity during treatment. The classical Caputo model keeps radiation effort high across time. The weighted-memory model, on the other hand, gives a smoother and lower control level. This means the new model can reduce tumor size with less radiation effort, making treatment easier and safer.

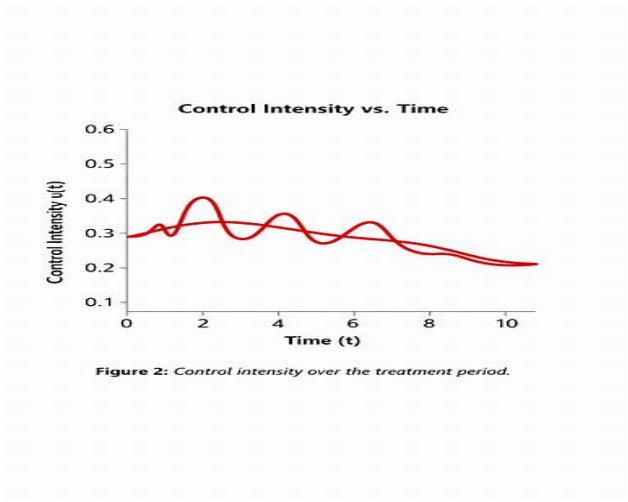

Fig. 2: Control intensity over the treatment period.

Toxicity Index

Fig. 3 shows the toxicity index. The classical Caputo model has the highest toxicity. The Caputo–Fabrizio model reduces it somewhat. The weighted-memory model gives the lowest values. This means our method not only reduces tumor size but also lowers side effects, making treatment safer.

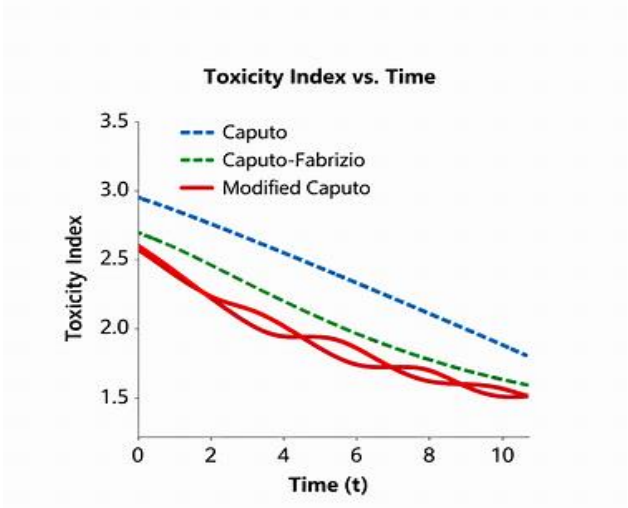

Fig. 3: Toxicity index under different fractional modeles.

4. Conclusion

The results and graphs show clear differences between the models. The classical Caputo (Caputo, 1967) reduces tumor size but often causes higher toxicity because the dose is not well balanced. The Caputo–Fabrizio (Caputo & Fabrizio, 2015), improves stability with its non-singular kernel, but it does not include patient history. The new Modified Caputo with weighted memory gives stronger tumor reduction and lower toxicity. This comes from the weighted memory function and better dose adjustment. **Fig. 1–3** clearly support these findings. Tumor size goes down faster, the control intensity changes smoothly, and the toxicity index stays lower than in the classical Caputo and Caputo–Fabrizio models. These results add to earlier work in fractional dynamics (Li & Chen, 2025) by making the mathematics closer to clinical practice. In conclusion, the weighted-memory operator that we used improves fractional calculus and makes it more useful for cancer modeling. It offers a path toward personalized therapy. Still, this study is limited to numerical tests. More work with clinical data and experiments is needed. Future studies should extend the operator to more fractional equations, test stability with other methods, and apply it in areas like population dynamics and control theory. This will make the method stronger and expand its use beyond cancer therapy, linking fractional calculus with real biomedical and engineering systems.

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6. Conflicts of Interest

The author declares no conflict of interest.

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